

# SO(3)-STRUCTURES ON 8-MANIFOLDS

SIMON G. CHIOSSI<sup>1)</sup>, ÓSCAR MACÍÁ<sup>1,2)</sup>

**ABSTRACT.** We study Riemannian 8-manifolds with an infinitesimal action of  $SO(3)$  by which each tangent space breaks into irreducible spaces of dimensions 3 and 5. The relationship with quaternionic, almost product- and  $PSU(3)$ -geometry is thoroughly explained using representation-theoretical arguments.

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## 1. INTRODUCTION

The purpose of this article is to study geometric structures on eight-dimensional Riemannian manifolds admitting a special infinitesimal action of the Lie group  $SO(3)$ . Our study represents the first flicker of the unfolding of a larger theory, as we shall have time to argue; it follows a long trail initiated by A.Gray and L.Hervella [14], who recognised the intrinsic torsion as the killer app to interpret, and handle, Riemannian manifolds  $\{M^n, g\}$  whose principal bundle of oriented orthonormal frames reduces to a bundle with structure group  $G \subset SO(n)$ , aka  $G$ -structures.

Among the flurry of  $G$ -structures considered ever since – for a comprehensive survey of which the adoption of [1] is recommended – we are concerned with those for which  $G = SO(3)$ . This Lie group can act reducibly on the tangent spaces of a manifold, a situation sorted out years ago by E.Thomas [33] and M.Atiyah [3], or irreducibly, in which case the general stance can be found in [11]. The first concrete results about an  $SO(3)$ -irreducible action were obtained in [6] and [10] on manifolds of dimension 5. The former

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represented a true breakthrough, for M.Bobiński and P.Nurowski proved that the rôle of the intrinsic torsion of an irreducible  $\mathrm{SO}(3)$ -manifold is played by a tensor defined naturally by the representations. Further advancement was made by I.Agricola et al. [2], in relationship to the topological obstructions to existence in dimension five in particular.

As a matter of fact  $\mathrm{SO}(3)$  is a quotient of the Lie group  $\mathrm{SU}(2)$ , and as such plays a role, in real dimensions  $4k$ , analogous to that of the circle group  $\mathrm{U}(1)$  in complex geometry. This paper aims at describing  $\mathrm{SO}(3)$ -structures on Riemannian manifolds  $\{M^8, g\}$  for which each tangent space decomposes in two summands

$$T_p M^8 = V \oplus W,$$

of dimensions three and five, each of which is irreducible under  $\mathrm{SO}(3)$ . This is achieved by fixing a representation of  $\mathrm{SO}(3)$  on  $\mathbb{R}^8$ , whose image we call  $\mathcal{G}$ . The reason for concentrating on dimension eight and focusing on that particular action is that

$$\mathcal{G} = \mathrm{SO}(3)$$

is contained in other intermediate Lie subgroups  $G$  of  $\mathrm{SO}(8)$  of a certain interest, namely

$$\mathrm{SO}(3) \times \mathrm{SO}(5), \quad \mathrm{PSU}(3), \quad \mathrm{Sp}(2)\mathrm{Sp}(1).$$

Each group of this triad is known to give rise to exciting geometric properties, see [25], [18, 34] and [28, 31] respectively, and is remarkably obtained for free when imposing the  $\mathcal{G}$ -action. What kickstarted the present work was the realisation that there is a deep relationship between  $\mathcal{G}$  and almost quaternion-Hermitian geometry, as noticed also by A.Gambioli [12]. The article [21], to which this is a sequel of sorts, made this idea concrete by studying the quaternionic geometry of  $\mathrm{SU}(3)$ , focusing in particular on the class of so-called nearly quaternionic structures, the closest quaternionic kin to non-complex nearly Kähler manifolds. Whereas the customary approach would consider a Lie group's bi-invariant metric properties, in that case there is one action of  $\mathrm{SU}(3)$  on the left, and another action of  $\mathrm{SO}(3)$  on the right.

One important point is to show explicitly how the ‘subordinate’  $G$ -structures relate to one another, and especially how they determine a  $\mathcal{G}$ -action. To this end we prove in Theorem 3.1 that not only  $\mathcal{G}$  is the triple intersection of the aforementioned groups, as expected, but one can recover it by using just two of the three. This result is rephrased with more algebraic flavour by Theorem 3.5, according to which, if we let  $\mathfrak{g} \subset \mathfrak{so}(8)$  be the Lie algebra of  $\mathcal{G}$ , then

$$\mathfrak{g}^\perp = \frac{\mathfrak{so}(3) \oplus \mathfrak{so}(5)}{\mathfrak{g}} + \frac{\mathfrak{psu}(3)}{\mathfrak{g}} + \frac{\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)}{\mathfrak{g}},$$

and the sum of any two quotients is isomorphic to the orthogonal complement of the third algebra.

Triality, a distinctive feature of dimension 8, manifests itself spectacularly by endowing the space  $\mathrm{SO}(8)/\mathrm{PSU}(3)$ , parametrising reductions of the structure group of  $TM^8$  to  $\mathrm{PSU}(3)$ , with the structure of a 3-symmetric space (Remark 3.1).

After a brief diversion on topology (section 4), which serves the purpose of finding obstructions to the existence of  $\mathcal{G}$ -actions, we dwell into the theory of  $G$ -structures. We determine the intrinsic-torsion space for  $\mathcal{G}$  by decomposing it into irreducible  $\mathcal{G}$ -modules

(Proposition 5.1); the core observation is that, of its 200 dimensions, exactly 3 arise from  $\mathcal{G}$ -invariant tensors that we can describe explicitly. We then compare this decomposition with the spaces of G-intrinsic torsion relative to the common subgroup  $\mathcal{G}$ , culminating in Proposition 5.4.

A further refinement is presented in the last section 6. There are six exterior forms invariant under  $\mathcal{G}$ , two 3-forms, two 4-forms and two 5-forms, defining the reduction. If one restricts to the case in which the intrinsic torsion is  $\mathcal{G}$ -invariant, Theorem 6.2 guarantees these three pairs will fit into an invariant deRham complex

$$(\Lambda^3 T^*M)^{\mathcal{G}} \xrightarrow{d} (\Lambda^4 T^*M)^{\mathcal{G}} \xrightarrow{d} (\Lambda^5 T^*M)^{\mathcal{G}}.$$

that governs the geometry entirely.

We should emphasize that this research area is very much in its infancy. We have concentrated on the Lie-theoretical point of view embodied by the intrinsic torsion, and put little stress on other aspects that should deserve a separate treatment, such as higher-dimensional Riemannian SO(3)-manifolds.

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## 2. SO(3) AND SUBORDINATE STRUCTURES

We recall the salient points of Sp(1)-representations, since the chief technique for decomposing exterior forms employs the weights of the action of  $\mathfrak{sp}(1) \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$ . Let  $H$  be the basic representation of Sp(1) on  $\mathbb{C}^2 = \mathbb{H}$  given by left-multiplying column vectors by matrices. The  $n$ th symmetric power of  $H$  is an irreducible representation of  $\mathbb{C}^{2n} = \mathbb{H}^n$  written

$$S^n = S^n(H).$$

Every irreducible Sp(1)-module is of this form for some nonnegative  $n \in \mathbb{Z}$ , and is the eigenspace of the Casimir operator with eigenvalue  $-n(n+2)$ . It can be decomposed into weight spaces of dimension one under the action of the Cartan subalgebra  $\mathfrak{sl}(2, \mathbb{C})$ , making  $S^n$  the unique irreducible representation with highest weight  $n$ . Since the weight is a homomorphism from tensor products of representations to the additive integers, weight-space decompositions can be used to determine tensor-, symmetric and skew-symmetric products of modules, as in the Clebsch-Gordan formula

$$S^n \otimes S^m \cong S^{n+m} \oplus S^{n+m-2} \oplus \dots \oplus S^{n-m+2} \oplus S^{n-m}$$

when  $n \geq m$ .

The Lie group SO(3) is the  $\mathbb{Z}_2$ -quotient of the simply connected covering Sp(1), and as such its complex representations coincide with the aforementioned ones. Thus by identifying SO(3)-modules with the above  $S^\lambda$  endowed with a real structure, we will allow ourselves to switch tacitly from complex to real throughout this work. For example  $V \cong \mathbb{R}^3$ , the fundamental representation of SO(3), will be considered the same as  $S^2 H$ , and the traceless symmetric product  $S_0^2 V \cong S^4 H$  will have dimension five.

A natural way to manufacture an  $\mathrm{SO}(3)$ -structure on a Riemannian manifold  $\{M^8, g\}$  is to fix a homomorphism

$$(2.1) \quad \mathrm{SO}(3) \longrightarrow \mathrm{SO}(8),$$

whose image will be denoted  $\mathcal{G}$ , that breaks the tangent space at each point  $p$  of  $M$  into three- and five-dimensional summands

$$(2.2) \quad T_p M^8 = V \oplus S_0^2 V = V \oplus W.$$

Note that  $W$  is in fact the irreducible  $\mathrm{SO}(3)$ -module used in [6, 10, 11] to study  $\mathrm{SO}(3)$ -manifolds of dimension 5 modelled on the Riemannian symmetric space  $\mathrm{SU}(3)/\mathrm{SO}(3)$ . The story there went as follows: there is a subgroup inside  $\mathrm{SO}(5)$  isomorphic to  $\mathrm{SO}(3)$  acting in an irreducible fashion, by which  $\mathfrak{su}(3) = \mathfrak{so}(3) + iW$ , and  $W \cong \mathbb{R}^5$  is identified with the space of symmetric and trace-free  $3 \times 3$  matrices  $S_0^2 \mathbb{R}^3$ . Although the  $\mathcal{G}$ -action prescribed by (2.2) is certainly not irreducible, it is the unique action of  $\mathrm{SO}(3)$  that decomposes  $\mathbb{R}^8$  into non-trivial  $\mathcal{G}$ -irreducible subspaces, and as such it is worth studying.

A more interesting way to view the splitting is to rewrite it as

$$(2.3) \quad T_p M^8 \otimes_{\mathbb{R}} \mathbb{C} \cong S^2 H \oplus S^4 H.$$

If a Riemannian 8-manifold admits such an infinitesimal action of  $\mathrm{SO}(3)$ , the endomorphisms of the complexified tangent bundle  $TM_c^* \otimes TM_c$  include a factor  $S^2$ , which is precisely the Lie algebra  $\mathfrak{g} \cong \mathfrak{so}(3)$  of  $\mathcal{G}$ .

There is a second reason for concentrating on dimension eight, and focusing on the particular action described by (2.3). The embedding of  $\mathrm{SO}(3)$  actually factors through other Lie groups of interest, refining (2.1) to this diagramme:

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{SO}(8) \\ & \searrow & \nearrow \\ & G & \end{array}$$

where  $G$  will be one of the following subgroups of  $\mathrm{SO}(8)$ :

$$(2.4) \quad \mathrm{Sp}(1)\mathrm{Sp}(2) = \mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(2), \quad \mathrm{PSU}(3) = \mathrm{SU}(3)/\mathbb{Z}_3, \quad \mathrm{SO}(3) \times \mathrm{SO}(5).$$

This enables us to understand the mutual relationship between an  $\mathrm{SO}(3)$ -structure and any of these  $G$ -structures. The main point to this section is to prove

**Proposition 2.1.** *A  $\mathcal{G}$ -structure (2.3) on a Riemannian eight-manifold  $\{M^8, g\}$  induces altogether an almost product structure, a  $\mathrm{PSU}(3)$ -structure and an almost quaternion-Hermitian structure.*

*Proof.* The argument will be broken up in cases corresponding to the above Lie groups  $G$ .

- (i)  $G = \mathrm{Sp}(2)\mathrm{Sp}(1)$ . What we will actually prove here is the intermediate ‘diagonal-type’ inclusion

$$\mathcal{G} \subset \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \subset \mathrm{Sp}(2)\mathrm{Sp}(1),$$

whereby the first factor is the irreducible  $\mathrm{Sp}(1)$  inside  $\mathrm{Sp}(2)$  (recall the latter is the universal cover of  $\mathrm{SO}(5)$ ), while the second is just the identity map's image. Similar  $\pm$  labelling will be used to identify the representations.

At each point of  $M$  the complexified tangent space  $T_c = S^2 \oplus S^4$  is quaternionic, since the action of  $\mathrm{Sp}(1)$  on the fundamental  $\mathrm{Sp}(2)$ -representation  $E \cong \mathbb{C}^4$  with highest weight  $(1, 0)$  ensures  $E = S^3 H$ . This means that (2.3) is isomorphic to

$$T_c \cong S_+^3 H \otimes H = E \otimes H,$$

if one takes, by convention,  $H = S_-^1 H$ . The orthogonal Lie algebra decomposes under  $\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$  as

$$(2.5) \quad \begin{aligned} \mathfrak{so}(8, \mathbb{C}) = \Lambda^2(S_+^3 S_-^1) &= (S_+^6 \oplus S_+^2) \oplus (\Lambda_0^2(S_+^3) \otimes S^2(S_-^1)) \oplus S_-^2 \\ &= \mathfrak{sp}(2)_+ \oplus (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^\perp \oplus \mathfrak{sp}(1)_-. \end{aligned}$$

The group  $\mathcal{G}$  then sits inside  $\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$  and its Lie algebra  $\mathfrak{g}$  corresponds to an  $S^2$ -module in  $S_+^2 \oplus S_-^2 = \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$ .

- (ii)  $G = \mathrm{PSU}(3)$ . The corresponding Lie algebra  $\mathfrak{psu}(3) = \mathfrak{su}(3)$  splits naturally as  $S^2 \oplus S^4$  if we think of a  $3 \times 3$  Hermitian matrix as the Cartan sum of a skew-symmetric matrix (in  $S^2 = \mathfrak{so}(3)$ ) and a purely imaginary symmetric matrix with no trace (whence  $W = S^4$ ).
- (iii)  $G = \mathrm{SO}(3) \times \mathrm{SO}(5)$ . The argument is formally the same of case (i), by virtue of the universal covering  $\mathrm{Sp}(1) \times \mathrm{Sp}(2) \xrightarrow{4:1} \mathrm{SO}(3) \times \mathrm{SO}(5)$ . In two words, the block-diagonal embedding  $\mathrm{SO}(3) \times \mathrm{SO}(5) \subset \mathrm{SO}(8)$  decomposes  $\mathfrak{so}(8) \cong \Lambda^2(W \oplus V) = \mathfrak{so}(5)_W \oplus (W \otimes V) \oplus \mathfrak{so}(3)_V$ , reflected in (2.5). qed

**Example 2.2.** *A class of  $\mathcal{G}$ -structures on  $M^8 = \mathrm{SU}(3)$ , giving a non-integrable analogue of quaternion-Kähler geometry, was studied in [21]. The action of  $\mathfrak{g}$  on  $\mathfrak{su}(3)$  described there spawned an almost quaternion-Hermitian structure of class  $\mathcal{W}_1^{AQH} \subset \mathcal{W}_{1+4}^{AQH}$  in the Cabrera-Swann terminology; the fact that the local Kähler forms generated a differential ideal with coefficients sitting in a traceless, symmetric matrix of 1-forms, was the main reason to start the study of  $\mathcal{G}$ -manifolds.*

Before passing to the next section, a few credits to complete the overall picture.

(i) Recall  $\mathrm{Sp}(2)\mathrm{Sp}(1)$  is one of Berger's holonomy groups [4], see [20, ch. 5] for an account of the theory. To S.Salamon [29] we owe the 'EH' formalism used throughout the paper, while for the exhaustive description of the intrinsic torsion of  $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -geometry one should refer to [30, 23].

(ii)  $\mathrm{PSU}(3)$ -structures were borne in on N.Hitchin's programme on special geometry [18] and were thoroughly explored by F.Witt [34]. The latter and Puhle's article [26] give an accurate description of the intrinsic torsion of  $\mathrm{PSU}(3)$ -structures, and our results are meant to complement those and enable to gain solid, intuitive understanding of the matter.

(iii) The classification of almost product structures begun with the work of A.Naveira [25]; deep geometric consequences and many examples were discussed by O.Gil-Medrano [13] and V.Miquel [24].

## 3. INTERSECTION THEOREM

Having shown that  $\mathrm{SO}(3)$  is a subgroup of all three of  $\mathrm{SO}(3) \times \mathrm{SO}(5)$ ,  $\mathrm{Sp}(2)\mathrm{Sp}(1)$ ,  $\mathrm{PSU}(3)$ , we know in particular it lies in their intersection. The threefold aspect of the theory is pivotal for understanding this and many other features, and the aim of this section is to prove that there is some redundancy: in other words, any two are enough to guarantee the retrieval of  $\mathcal{G} = \mathrm{SO}(3)$ . More formally,

**Theorem 3.1.** *Let  $\mathcal{G} \cong \mathrm{SO}(3)$  be the subgroup of  $\mathrm{SO}(8)$  acting infinitesimally on a Riemannian 8-manifold  $\{M, g\}$  by decomposing tangent spaces like*

$$T_p M = V \oplus S_0^2 V,$$

where  $V \cong \mathbb{R}^3$  is the fundamental representation. Then

- (1)  $\mathcal{G} = (\mathrm{SO}(3) \times \mathrm{SO}(5)) \cap \mathrm{PSU}(3)$ ,
- (2)  $\mathcal{G} = \mathrm{PSU}(3) \cap \mathrm{Sp}(2)\mathrm{Sp}(1)$ ,
- (3)  $\mathcal{G} = \mathrm{Sp}(2)\mathrm{Sp}(1) \cap (\mathrm{SO}(3) \times \mathrm{SO}(5))$ .

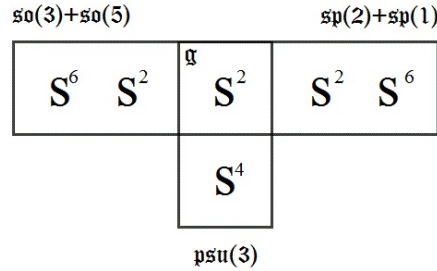


FIGURE 1.  $\mathfrak{so}(8)$  as the sum of  $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ ,  $\mathfrak{psu}(3)$ ,  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ , neatly intersecting in  $\mathfrak{g}$ .

The proof, sketched in fig. 1, consists of the ensuing series of lemmas and is minutely carried out at the level of the corresponding Lie algebras.

**Lemma 3.2.** *Retaining the notation of (2.2),*

$$\mathfrak{g} = (\mathfrak{so}(3)_V \oplus \mathfrak{so}(5)_W) \cap \mathfrak{su}(3).$$

*Proof.* The right-hand-side intersection, say  $\mathfrak{g}$ , contains the diagonal  $\mathfrak{g}$ . Now,  $\mathfrak{g}$  is sitting diagonally in  $\mathfrak{so}(3)_V \oplus \mathfrak{so}(5)_W$ , so the combination  $\mathfrak{g} \oplus \mathfrak{so}(3)_V$  would generate  $\mathfrak{so}(5)_W$  if  $\mathfrak{su}(3)$  were to contain  $\mathfrak{g}$  together with  $\mathfrak{so}(3)_V$ . Since  $\mathfrak{so}(5)_W$  is not a subalgebra of  $\mathfrak{su}(3)$ , we have  $\mathfrak{g} = \mathfrak{g}$ . qed

**Lemma 3.3.** *We have*

$$\mathfrak{g} = \mathfrak{su}(3) \cap (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)).$$

*Proof.* Let  $\mathfrak{g}$  be the intersection on the right, and recall  $\mathfrak{g} \subseteq \mathfrak{g}$ . Moreover,  $\mathfrak{g}$  sits diagonally in  $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ . The fundamental representation  $\mathbb{C}^3$  of  $\mathfrak{su}(3)$  corresponds either to  $\mathbb{C}^3 = S^2$  or  $\mathbb{C}^3 = H \oplus \mathbb{C}$ , if viewed as a  $\mathfrak{g}$ -module. But since  $\mathfrak{su}(3)$  is the space of traceless endomorphisms of  $\mathbb{C}^3$ , the two instances regard  $\mathrm{End}_0(S^2) = S^4 \oplus S^2$  or

$\text{End}_0(\mathbb{H} \oplus \mathbb{C}) = \mathbb{S}^2 \oplus 2\mathbb{H} \oplus \mathbb{C}$  respectively. Neither contains a  $3+3$ -dimensional submodule that can be identified with  $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$ , so  $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- \not\subseteq \mathfrak{su}(3)$  and the  $\mathbb{S}^2$  factor of the first instance must coincide with  $\mathfrak{g}$ . qed

**Lemma 3.4.** *In the notation of (2.2),*

$$\mathfrak{g} = (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)) \cap (\mathfrak{so}(3)_V \oplus \mathfrak{so}(5)_W).$$

*Proof.* Let  $\mathfrak{g}$  denote the intersection, as usual. As we know that  $\mathfrak{g} \subset \mathfrak{g}$ , let us assume by contradiction  $\mathfrak{g} \neq \mathfrak{g}$ .

The following possibilities arise when looking at  $\mathfrak{g} \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ : either (i)  $\mathfrak{sp}(1) \subseteq \mathfrak{g}$ , or (ii)  $\mathfrak{sp}(1) \not\subseteq \mathfrak{g}$ .

Case (i):  $\mathfrak{g} = \mathfrak{sp}(1) \oplus \mathfrak{k}$ , with  $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{sp}(2)$ . Let  $G$  and  $K$  denote the connected Lie groups of  $\mathfrak{g}$ ,  $\mathfrak{k}$ . Complex tangent spaces decompose as  $T_c = \mathbb{E}H$ , and as  $\mathfrak{sp}(1) \subseteq \mathfrak{g}$ , the  $\text{Sp}(1)$ -irreducible representation  $H$  is irreducible also under  $G$ . Similarly  $E$  can be seen as a  $K$ -module, since  $\mathfrak{k} \subseteq \mathfrak{sp}(2)$ ; in contrast to  $H$ , however, it can be either reducible or irreducible, depending on whether  $\mathfrak{k} \subset \mathfrak{sp}(1)_+ \subset \mathfrak{sp}(2)$  (case (i.i)) or  $\mathfrak{sp}(1)_+ \subseteq \mathfrak{k} \subseteq \mathfrak{sp}(2)$  (case (i.ii)).

(i.i): write  $E = \oplus_i A_i$  as a sum of irreducible  $K$ -modules, so that  $T_c = (\oplus_i A_i) \otimes H = \oplus_i (A_i \otimes H)$  is a sum of  $G$ -irreducible terms of even dimension. The latter fact clashes with the dimensions of  $\mathbb{R}^5 \oplus \mathbb{R}^3$  arising from  $\mathfrak{g} \subset \mathfrak{so}(5) \oplus \mathfrak{so}(3)$ .

(i.ii):  $E$  is  $K$ -irreducible, making  $T_c = \mathbb{E}H$  irreducible under  $G$ . But again, (2.3) contradicts irreducibility.

Case (ii): as  $\mathfrak{sp}(1) \not\subseteq \mathfrak{g}$ , we write  $\mathfrak{g} \subseteq \mathfrak{k} \oplus \mathfrak{sp}(1)$ , where now  $\mathfrak{k}$  is the projection of  $\mathfrak{g}$  to  $\mathfrak{sp}(2)$ . But since  $\mathfrak{g}$  acts diagonally,  $\mathfrak{sp}(1)_+ \subseteq \mathfrak{k} \subseteq \mathfrak{sp}(2)$ , and  $\mathfrak{k}$  is a  $\text{Sp}(1)_+$ -representation inside  $\mathfrak{sp}(2)$ . Considering the decomposition of  $\mathfrak{sp}(2) = \mathbb{S}_+^6 \oplus \mathbb{S}_+^2$  in irreducible  $\text{Sp}(1)_+$ -modules, with  $\mathbb{S}_+^2 \cong \mathfrak{sp}(1)_+$ , we face another dichotomy: either (ii.i)  $\mathfrak{k} = \mathfrak{sp}(1)_+$ , implying that  $\mathfrak{g} \oplus \mathfrak{sp}(1)_+ \subseteq \mathfrak{g}$ , or (ii.ii)  $\mathfrak{k} = \mathbb{S}_+^6$  and  $\mathfrak{g} \oplus \mathbb{S}_+^6 \subseteq \mathfrak{g}$ .

(ii.i): as  $\mathfrak{g}$  sits diagonally in  $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-$ , the subalgebra  $\mathfrak{g} \oplus \mathfrak{sp}(1)_+ \subseteq \mathfrak{g}$  would detect an  $\mathfrak{sp}(1)_- = \mathfrak{sp}(1)$  inside  $\mathfrak{g}$ , against the general hypothesis (ii).

(ii.ii):  $\mathbb{S}_+^6$  is not a subalgebra of  $\mathfrak{sp}(2)$ , so the Lie bracket of  $\mathfrak{g} \oplus \mathbb{S}_+^6 \subseteq \mathfrak{g}$  would produce an  $\mathfrak{sp}(2)$ -term inside  $\mathfrak{g}$ . By the same argument as before, the diagonal  $\mathfrak{g} \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$  would make  $\mathfrak{g} \oplus \mathfrak{sp}(2)$  generate an  $\mathfrak{sp}(1)$  in  $\mathfrak{g}$ , again against (ii).

Both (i), (ii) disproving the initial assumption, we conclude that  $\mathfrak{g} = \mathfrak{g}$ . qed

We can rephrase Theorem 3.1 in a perhaps-more-eloquent fashion

**Theorem 3.5.** *Let  $\mathfrak{g}_i$ ,  $i = 1, 2, 3$ , denote the Lie algebras of the groups  $\text{SO}(3) \times \text{SO}(5)$ ,  $\text{PSU}(3)$ ,  $\text{Sp}(2)\text{Sp}(1)$ ,  $\mathfrak{g}_i^\perp$  the complements in  $\mathfrak{so}(8)$  and  $\mathfrak{g}$  the Lie algebra of  $\mathcal{G} = \text{SO}(3)$ . Then*

$$\begin{aligned} \mathfrak{g}_i^\perp &= (\mathfrak{g}_j/\mathfrak{g}) \oplus (\mathfrak{g}_k/\mathfrak{g}), \quad i \neq j \neq k = 1, 2, 3 \\ \mathfrak{g}^\perp &= \bigoplus_{i=1}^3 (\mathfrak{g}_i/\mathfrak{g}). \end{aligned}$$

*Proof.* As  $\mathfrak{so}(8) = \mathfrak{g}_i \oplus \mathfrak{g}_i^\perp$  and  $\mathfrak{g}_i = \mathfrak{g} \oplus (\mathfrak{g}_i/\mathfrak{g})$ , the assertion is a straightforward consequence of a dimension count plus  $\mathfrak{g}_i \cap \mathfrak{g}_j = \mathfrak{g}$  (Theorem 3.1). qed

For the sake of clarity, and for later use, here are the explicit modules involved. Start from the irreducible  $\mathcal{G}$ -decomposition

$$(3.1) \quad \mathfrak{so}(8) \cong \Lambda^2(\mathbb{S}^2 \oplus \mathbb{S}^4) = 2\mathbb{S}^6 \oplus \mathbb{S}^4 \oplus 3\mathbb{S}^2 \cong \mathfrak{g} \oplus (2\mathbb{S}^6 \oplus \mathbb{S}^4 \oplus 2\mathbb{S}^2),$$

where  $\mathfrak{g} \cong \mathbb{S}^2$ ,  $\mathfrak{g}^\perp = (2\mathbb{S}^6 \oplus \mathbb{S}^4 \oplus 2\mathbb{S}^2)$ . Read in ‘VW’ terms, that tells  $(\mathfrak{so}(3)_V \oplus \mathfrak{so}(5)_W)^\perp \cong \mathbb{S}_V^2 \otimes \mathbb{S}_W^4$  as  $\mathrm{SO}(3)_V \times \mathrm{SO}(3)_W$ -modules. Reducing to  $\mathcal{G}$ -modules by taking the diagonal embedding (forgetting where the terms come from, ie dropping the subscripts) leads to

$$(\mathfrak{so}(3) \oplus \mathfrak{so}(5))^\perp \cong \mathbb{S}^6 \oplus \mathbb{S}^4 \oplus \mathbb{S}^2$$

hence  $(\mathfrak{so}(3) \oplus \mathfrak{so}(5))/\mathfrak{g} = \mathbb{S}^2 \oplus \mathbb{S}^6$ .

Similarly, starting from (2.5) and identifying  $\mathbb{S}_+^\lambda \cong \mathbb{S}_-^\lambda$ , we obtain

$$(\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^\perp \cong (\Lambda_0^2(\mathbb{S}^3) \otimes \mathbb{S}^2(\mathbb{S}^1)) \cong \mathbb{S}^4 \otimes \mathbb{S}^2 = \mathbb{S}^6 \oplus \mathbb{S}^4 \oplus \mathbb{S}^2$$

with  $(\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))/\mathfrak{g} = \mathbb{S}^6 \oplus \mathbb{S}^2$ .

Finally, it is well documented [21] that the adjoint representation of  $\mathfrak{su}(3)$  in  $\mathfrak{so}(8)$ , decomposed under  $\mathcal{G}$ , coincides with  $\mathbb{S}^2 \oplus \mathbb{S}^4$ . Hence

$$(3.2) \quad \mathfrak{su}(3)^\perp = 2\mathbb{S}^6 \oplus 2\mathbb{S}^2$$

and  $\mathfrak{su}(3)/\mathfrak{g} \cong \mathbb{S}^4$ .

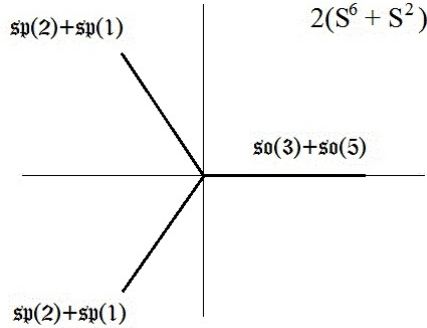


FIGURE 2. The space  $2(\mathbb{S}^2 \oplus \mathbb{S}^6)$  rotates by  $2\pi/3$  incarnating at each turn a pair of subalgebras (bold lines), while  $\mathfrak{psu}(3)$  remains fixed (centre of symmetry).

*Remark 3.1* ([34]). The triality automorphism of  $\mathbb{R}^8$ , in some loose sense, plays a similar role to that of self-duality in dimension 4. Its stabiliser is  $G_2 \subset \mathrm{Spin}(8)$ , and although the exceptional Lie group does not contain  $\mathcal{G}$ , but rather a diagonal  $\mathrm{SO}(3)$  embedded in  $\mathrm{SO}(4) \subset G_2$ , the decomposition of  $\mathfrak{so}(8, \mathbb{C})$  detects the existence of an automorphism of order three (fig. 2) that permutes the three  $\mathbb{S}^2$  by rotating the two copies of  $\mathbb{S}^6 \oplus \mathbb{S}^2$  by  $2\pi/3$ . This automorphism has  $\mathrm{PSU}(3) \times \mathbb{Z}_2$  as fixed-point set in  $\mathrm{Spin}(8)$  (and  $\mathrm{PSU}(3)$  in  $\mathrm{SO}(8)$ ).



The 20-dimensional manifold  $\mathrm{SO}(8)/\mathrm{PSU}(3)$  is a 3-symmetric space [15] whose tangent space at each point is isomorphic to  $\mathrm{S}^2\mathbb{C}^3 \oplus \mathrm{S}^2\overline{\mathbb{C}}^3$ . This is given by (3.2) and is real-irreducible under  $\mathrm{PSU}(3)$ , whereas over  $\mathrm{SO}(3)$  it can be decomposed as a sum  $(\mathrm{S}^2 \oplus \mathrm{S}^6) \oplus f_*(\mathrm{S}^2 \oplus \mathrm{S}^6)$ , where  $f$  is the induced isometry of order three. If the sum of the first two terms corresponds to  $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ , then  $f_*(\mathrm{S}^2 \oplus \mathrm{S}^6)$  is  $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ . The third space required to ‘visualise’ triality is the image of the square  $f_* \circ f_*$ , and is only apparently missing. It is another quaternionic subalgebra fixed by an ‘anti-selfdual’ four-form [21], and we will return to it later (see (6.4)).

#### 4. TOPOLOGICAL OBSERVATIONS

This section marks a slight detour intended to clarify aspects of compact  $\mathcal{G}$ -manifolds of dimension 8. Some pieces of information can be extracted from work of M. Čadež et al, see [8, 7] for instance, and traced back to earlier papers [32, 16], although a broader standpoint will have to await further study.

There several ways to explain why, almost automatically,

**Proposition 4.1.** *An oriented  $\mathcal{G}$ -manifold  $M^8$  is spin.*

*Proof.* Let us see three easy arguments to corroborate once more the theory’s richness. First, the Stiefel-Whitney class  $w_2(M) \in H^2(M, \mathbb{Z}_2)$  is twice the Marchiafava-Romani class [22] for  $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structures, and hence null mod two.

Secondly, a quaternion-Hermitian structure renders the 8-dimensional modules  $\mathrm{S}^2\mathrm{H} \oplus \Lambda_0^3\mathrm{E}$  and  $\mathrm{T}_c = \mathrm{EH}$  real and  $\mathrm{SO}(8)$ -invariant; these must therefore factor through  $\mathrm{Sp}(2)\mathrm{Sp}(1)$ .

Thirdly,  $\mathrm{PSU}(3)$ -structures are spin and the irreducible inclusion  $\mathcal{G} \subset \mathrm{SO}(5)$  lifts to  $\mathrm{Sp}(2) = \mathrm{Spin}(5) \subset \mathrm{Spin}(8)$ ; in fact, the essence of Lemma 3.3 is precisely that the fibre of this covering does not interfere with the discrete centre of  $\mathrm{SU}(3)$ . qed

Instead of relying on Wu’s formula to compute other Stiefel-Whitney classes in the  $\mathbb{Z}_2$ -cohomology ring, we shall exploit the crucial inclusion  $\mathcal{G} \subset \mathrm{PSU}(3)$ , and from [34] we know all Stiefel-Whitney classes vanish, except possibly for  $w_4(M)$  which squares to zero, at any rate.

Starting from decomposition (2.3) we determine the integral Pontrjagin classes  $p_i(M) \in H^{4i}(M, \mathbb{Z})$ ,  $i = 1, 2$ .

**Proposition 4.2.** *The Pontrjagin classes of a  $\mathcal{G}$ -manifold  $\{M^8, g\}$  are related by*

$$\begin{aligned} 4p_2(M) &= p_1(M) \smile p_1(M) \\ p_1^2 &\in 8640 \mathbb{Z}. \end{aligned}$$

*Proof.* We are entitled to assume there is a circle acting on the tangent bundle by pulling  $\mathrm{T}_c$  back to some larger manifold fibring over  $M$ , if necessary (splitting principle). Thus we can decompose

$$(4.1) \quad V_c := V \otimes_{\mathbb{R}} \mathbb{C} = L \oplus \overline{L} \oplus \mathbb{C}$$

using a complex line bundle  $L$  with Chern character  $e^x$ . Recall  $\mathrm{S}_0^2 V_c = V_c^{\otimes 2} \ominus \Lambda^2 V_c \ominus \mathbb{C} \cong V_c^{\otimes 2} \ominus V_c \ominus \mathbb{C}$ , and that  $ch$  is a ring homomorphism from  $K_{\mathbb{C}}(M)$  to the even rational

cohomology, to the effect that

$$\begin{aligned} ch(T_c) &= ch(V_c) + ch(S_0^2 V_c) \\ &= 2 \cosh x + 1 + 2 \cosh 2x + 2 \cosh x + 1 \\ &= 8 + 6x^2 + 3x^4. \end{aligned}$$

Viewing the Chern classes of the holomorphic tangent bundle  $T^{1,0}$  as elementary symmetric polynomials  $s_j = \sum x_k^j$  in the variables  $x_1, \dots, x_4$  allows to factorise the total Chern class as  $c(T^{1,0}) = \prod (1 + x_k)$ . Thus we can write the Chern character  $ch(T^{1,0}) = \sum e^{x_k} = 4 + s_1 + \frac{1}{2}s_2 + \frac{1}{6}s_3 + \dots$ , whereby

$$ch(T_c)[M] = rank T_c + p_1 + \frac{1}{12}(p_1^2 - 2p_2),$$

since  $c_2 = -p_1$ ,  $c_4 = p_2$ , cf. [29]. By comparison we then have  $p_1(M) = 6x^2$ ,  $p_2(M) = 9x^4$ , and the first statement follows.

As for the second claim, it is possible to estimate the Pontrjagin number by invoking other characteristic classes. The Todd class  $Td(T_c) = \prod \frac{x_k}{1-e^{x_k}}$  equals  $1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 - p_1) - \frac{1}{24}c_1p_1 - \frac{1}{720}(c_1^4 + 4c_1^2p_1 - \frac{11}{4}p_1^2 - c_1c_3)$ , telling that the square  $p_1^2$  has to be a rather large integer, for it is divisible by 4·720 at least, irrespective of the vanishing of the first Chern class, for example. Results of [34] show that a compact oriented PSU(3)-manifold  $M^8$  satisfies  $p_1^2 \in 216\mathbb{Z}$ ; altogether, the lowest common multiple of the relevant factors is easily seen to equal 8640. qed

*Remark 4.1.* Alas, the possibility that  $p_1$  vanishes remains, and it would be valuable to know non-trivial examples with Dirac index equal zero.

The parallelisable manifold  $S^3 \times S^5$  (product of odd-dimensional spheres, here), despite having all classes zero, cannot be a  $\mathcal{G}$ -manifold: the five-sphere possesses no SO(3)-structure whatsoever, neither irreducible nor standard, due to the wrong values of Kervaire's semi-characteristics [11].

Still, it is interesting how the above is the best possible outcome of the Borel-Hirzebruch theory [17]: neither resorting to Riemann-Roch, nor computing the  $\hat{A}$ -genus, improve this estimate. Indeed, it is known that a compact spin manifold fulfilling equation (4.1) has signature  $\sigma = b_4^+ - b_4^-$  equal 16 times  $\hat{A}_2$ , so the Hirzebruch-Thom signature Theorem will imply  $60\sigma = p_1^2$ , wherefore

$$\hat{A}_2(M^8) = \frac{1}{16 \cdot 60} \int_M p_1^2(M^8).$$

At any rate, the existence of the quaternionic structure on  $M^8$  forces

$$8e + p_1^2 - 4p_2 = 0,$$

cf. [29, 9], so evidently:

**Corollary 4.3.** *A compact  $\mathcal{G}$ -manifold  $\{M^8, g\}$  has*

$$e(TM^8) = 0.$$

This is a useful obstruction, for it can prevent the existence of  $\mathcal{G}$ -structures.

**Examples 4.4.** *The Graßmannian  $SO(8)/(SO(3) \times SO(5))$  of real, oriented three-planes in  $\mathbb{R}^8$  fails the corollary, and as such it does not admit an infinitesimal  $\mathcal{G}$ -action of our type. As we know, in fact, the denominator embeds in  $SO(8)$  in the ‘wrong’ way.*

*The Wolf space  $G_2/SO(4)$  is a quaternion-Kähler manifold of positive scalar curvature, so its  $\hat{A}$ -genus is zero, whereas the Euler characteristic is not. That means it cannot carry a  $\mathcal{G}$ -structure, either.*

The corollary also falls out of the  $PSU(3)$ -side of the story, as we learn from [34].

The choice of (4.1) affects

$$(4.2) \quad S^4 \oplus S^2 = (L^2 \oplus \bar{L}^2 \oplus L \oplus \bar{L} \oplus \mathbb{C}) \oplus L \oplus \bar{L} \oplus \mathbb{C},$$

by singling out an almost complex structure. This comes from picking a point  $z$  in the fibre of the twistor fibration  $\mathbb{P}^1 \hookrightarrow \mathcal{Z} \rightarrow M$  ( $|z| = 1$  reduces  $SO(3)$  to  $U(1)$ , hence  $S^2 = \mathbb{C} \oplus \mathbb{R}$ ). The almost complex structure is defined by selecting a space of holomorphic tangent vector fields, and there should be a whole 2-parameter family thereof, depending on the choices of a line  $L = L' \cos \theta + L'' \sin \theta$ ,  $L' \in S^2$ ,  $L'' \in S^4$  and of a trivial term from a similar combination of the  $\mathbb{C}$ s in (4.2). By asserting that the space of  $(0, 1)$ -forms is annihilated by

$$T^{1,0} = L^2 + 2\bar{L} + \mathbb{C},$$

where  $L^{1/2} + L^{-1/2} = H$ , we are fixing  $J$ . This gives back (4.1), by the way.

There are other possible almost complex structures, two of which are fairly obvious: declaring  $L^2 + L + L + \mathbb{C}$  to be  $(1, 0)$ -vectors defines, say,  $J'$ , while  $L^2 + \bar{L} + L + \mathbb{C}$  gives  $J''$ . The latter is a quaternionic ‘twistor’ structure, because  $S^3 \otimes H = (L^{3/2} + L^{1/2} + L^{1/2} + L^{-3/2})L^{1/2} = T_{J''}^{1,0}$ . By contrast  $J'$ , already met in [21], is non-quaternionic as  $T_{J'}^{1,0} \neq E \otimes L^{1/2} = L^2 + L + \bar{L} + \mathbb{C}$ . There are three more basic almost complex structures obtained by complex conjugation, ie coming from swapping  $T^{1,0}$  and  $T^{0,1}$ .

## 5. RELATIVE INTRINSIC TORSION

We now begin to describe the intrinsic torsion for  $\mathcal{G}$ , explaining in particular how the torsion spaces of the subordinate  $G$ -structures relate to each other. Following [14] we view two-forms as traceless, skew-symmetric matrices, then decompose the space of intrinsic-torsion tensors

$$TM^8 \otimes \mathfrak{g}^\perp \subset TM^8 \otimes \Lambda^2 T^*M^8$$

into irreducible modules under the action of  $\mathcal{G}$  at each point. By equation (3.1) thus, we immediately have

**Proposition 5.1.** *The intrinsic torsion  $\tau_{\mathcal{G}}$  of the  $\mathcal{G}$ -structure is a tensor belonging in*

$$(S^2 \oplus S^4) \otimes \frac{\mathfrak{so}(8)}{\mathfrak{g}} = 2S^{10} \oplus 5S^8 \oplus 8S^6 \oplus 10S^4 \oplus 8S^2 \oplus 3\mathbb{R}.$$

*This space has dimension 200 and contains a 3-dimensional subspace of  $\mathcal{G}$ -invariant tensors.* *qed*

$\mathcal{G}$ -invariant subspaces will be the primary object of concern in section 6.

**Definition 5.2.** For any given Lie group  $G$  containing  $\mathcal{G}$  we denote by  $\tau_{\mathcal{G}}^G$  the intrinsic torsion of a  $G$ -structure decomposed under  $\mathcal{G}$ , and call it the  $G$ -torsion relative to  $\mathcal{G}$ , or just relative  $G$ -torsion,  $\mathcal{G}$  being implicit most of the times.

For simplicity, we disregard tensor-product signs and either juxtapose factors, or separate them by a full stop. The following lemma expresses the torsion spaces of  $G = \text{SO}(3) \times \text{SO}(5)$ -,  $\text{PSU}(3)$ -, and  $\text{Sp}(2)\text{Sp}(1)$ -structures in terms of  $\mathcal{G}$ -modules. The ensuing Proposition 5.4 will show, in the same spirit of section 3, how two among  $\tau_{\mathcal{G}}^{\text{SO}(3) \times \text{SO}(5)}$ ,  $\tau_{\mathcal{G}}^{\text{PSU}(3)}$  and  $\tau_{\mathcal{G}}^{\text{Sp}(2)\text{Sp}(1)}$  are enough to determine the third, and hence the  $\mathcal{G}$ -intrinsic torsion as well.

**Lemma 5.3.** Let  $G = \text{SO}(3) \times \text{SO}(5)$ ,  $\text{PSU}(3)$ ,  $\text{Sp}(2)\text{Sp}(1)$ . The relative  $G$ -torsion  $\tau_{\mathcal{G}}^G$  of  $\{M^8, g\}$  lives in the direct sum of the following modules:

	$S^{10}$	$S^8$	$S^6$	$S^4$	$S^2$	$\mathbb{R}$	$\dim_{\mathbb{R}}$
$\tau_{\mathcal{G}}^{\text{SO}(3) \times \text{SO}(5)}$	1	3	5	6	5	2	120
$\tau_{\mathcal{G}}^{\text{PSU}(3)}$	2	4	6	8	6	2	158
$\tau_{\mathcal{G}}^{\text{Sp}(2)\text{Sp}(1)}$	1	3	5	6	5	2	120

*Proof.*  $G = \text{SO}(3) \times \text{SO}(5)$ : The six components of the  $\text{SO}(3) \times \text{SO}(5)$ -torsion are

$$\tau_{\text{SO}(3) \times \text{SO}(5)} \in \Lambda^2 V.W \oplus S_0^2 V.W \oplus W \oplus V.\Lambda^2 W \oplus V.S_0^2 W \oplus V$$

if  $\text{TM} = V \oplus W$ , see [25]. The terms are ordered on purpose, so that the  $i$ th module corresponds to Naveira's  $i$ th (irreducible) class  $\mathcal{W}_i^{AP}$ . The by-now-customary identifications  $V \cong S_V^2, W \cong S_W^4$  gives the  $\text{SO}(3) \times \text{SO}(5)$ -torsion space decomposed relatively to the subgroup  $\text{SO}(3)_V \times \text{SO}(3)_W$

$$S_V^2 S_W^4 \oplus S_V^4 S_W^4 \oplus S_W^4 \oplus (S_V^2 S_W^6 \oplus S_V^2 S_W^4) \oplus (S_V^2 S_W^8 \oplus S_V^2 S_W^4) \oplus S_V^2$$

By taking  $\mathcal{G} = \text{SO}(3)_V = \text{SO}(3)_W$  and using Clebsch-Gordan we conclude.

$G = \text{PSU}(3)$ : From Theorem 3.5,  $\mathfrak{su}(3)^\perp = 2S^6 \oplus 2S^2$  as  $\mathcal{G}$ -modules, and the claim is immediate.

$G = \text{Sp}(2)\text{Sp}(1)$ : Using the standard quaternionic notation whereby  $\Lambda_0^2 E.E \ominus E$  is called  $K$ , the four terms in

$$T_c \otimes (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^\perp = E.S^3 H \oplus K.S^3 H \oplus K.H \oplus E.H$$

correspond to the four basic classes  $\mathcal{W}_i^{AQH}$  of [23]. Now identify  $H \cong S_-^1$  and  $E \cong S_+^3$ , and decompose under  $\text{Sp}(1)_+ \times \text{Sp}(1)_-$ , to the effect that the quaternionic torsion space relative to  $\text{Sp}(1)_+ \times \text{Sp}(1)_-$  reads

$$S_+^3 S_-^3 \oplus (S_+^7 S_-^3 \oplus S_+^5 S_-^3 \oplus S_+^1 S_-^3) \oplus (S_+^7 S_-^1 \oplus S_+^5 S_-^1 \oplus S_+^1 S_-^1) \oplus S_+^3 S_-^1.$$

The final step is the diagonal identification  $\mathrm{Sp}(1)_+ = \mathrm{Sp}(1)_-$ , that produces the required  $\mathcal{G}$ -modules. qed

*Remark 5.1.* The reader interested in a description of the irreducible modules of a general  $\mathrm{PSU}(3)$ -structure should consult [34, 26], while a detailed analysis of the almost quaternion-Hermitian case can be found in [23]. The result about the quaternionic group was proved in [21].

We shall treat in the next section the two-dimensional subspace of  $\mathcal{G}$ -invariant tensors common to all relative torsion spaces (cf. penultimate column in previous table).

For the last, summarising result of this part we need new labels: so let us write

$$\mathbf{P} = \mathrm{SO}(3) \times \mathrm{SO}(5), \quad \mathbf{R} = \mathrm{PSU}(3), \quad \mathbf{Q} = \mathrm{Sp}(2)\mathrm{Sp}(1),$$

and denote with  $\mathfrak{p}$ ,  $\mathfrak{r}$ ,  $\mathfrak{q}$  the corresponding Lie algebras. Theorem 3.5 says

$$\mathrm{T}^* \otimes \mathfrak{p}^\perp = \left( \mathrm{T}^* \otimes \frac{\mathfrak{r}}{\mathfrak{g}} \right) \oplus \left( \mathrm{T}^* \otimes \frac{\mathfrak{q}}{\mathfrak{g}} \right).$$

Now call  $\tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{R}) \in \mathrm{T}^* \otimes (\mathfrak{q}/\mathfrak{g})$  the component of  $\tau_{\mathcal{G}}^{\mathbf{P}}$  appearing in  $\tau_{\mathcal{G}}^{\mathbf{R}}$  but not present in  $\tau_{\mathcal{G}}^{\mathbf{Q}}$ , and similarly for  $\tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{Q}) \in \mathrm{T}^* \otimes (\mathfrak{r}/\mathfrak{g})$ . We can then write, informally,

$$\tau_{\mathcal{G}}^{\mathbf{P}} = \tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{R}) \oplus \tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{Q}).$$

A similar argument makes it easy to check that these components satisfy certain relations, for all permutations of  $\mathbf{P}, \mathbf{R}, \mathbf{Q}$ , as in

**Proposition 5.4.** *The tensor  $\tau_{\mathcal{G}}$  of  $\{\mathbf{M}^8, g\}$  determines  $\mathbf{P}$ -,  $\mathbf{Q}$ -,  $\mathbf{R}$ -structures whose relative torsion tensors  $\tau_{\mathcal{G}}^{\mathbf{P}}, \tau_{\mathcal{G}}^{\mathbf{Q}}, \tau_{\mathcal{G}}^{\mathbf{R}}$  satisfy the cyclic conditions*

$$\begin{aligned} \tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{R}) &= \tau_{\mathcal{G}}^{\mathbf{R}}(\mathbf{P}), \\ \tau_{\mathcal{G}}^{\mathbf{P}} &= \tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{R}) \oplus \tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{Q}), \\ \tau_{\mathcal{G}} &= \tau_{\mathcal{G}}^{\mathbf{P}}(\mathbf{R}) \oplus \tau_{\mathcal{G}}^{\mathbf{R}}(\mathbf{Q}) \oplus \tau_{\mathcal{G}}^{\mathbf{Q}}(\mathbf{P}). \end{aligned}$$

*In particular, any two yield the third.* qed

## 6. INVARIANT TORSION

It is worth remarking that  $\mathcal{G}$  stabilises certain exterior differential forms, as sanctioned by the ‘ $\mathbb{R}$ ’ terms in the previous table or by the singlets in

$$\begin{aligned} \Lambda^3 &\cong \mathbf{S}^8 \oplus 3\mathbf{S}^6 \oplus 3\mathbf{S}^4 \oplus 3\mathbf{S}^2 \oplus 2\mathbb{R} \cong \Lambda^5, \\ \Lambda^4 &\cong 2\mathbf{S}^8 \oplus 2\mathbf{S}^6 \oplus 6\mathbf{S}^4 \oplus 2\mathbf{S}^2 \oplus 2\mathbb{R}. \end{aligned}$$

These  $\mathcal{G}$ -invariant forms are two 3-forms  $\alpha, \beta$  and one 4-form  $\gamma$ , together with the Hodge duals  $*\gamma \in \Lambda^4$ ,  $*\alpha, *\beta \in \Lambda^5$ . Their algebraic nature can be described by tracking down the exact module they belong in:  $\alpha$  appears in the decomposition of  $\Lambda^3\mathbf{S}^2 \subset \Lambda^3$ , while its dual  $*\alpha$  shows up in  $\Lambda^5\mathbf{S}^4 \subset \Lambda^5$ . The form  $\beta$  spans the one-dimensional subspace in  $\Lambda^2\mathbf{S}^4 \otimes \mathbf{S}^2 \subset \Lambda^3$ , and  $*\beta$  lives in  $\Lambda^2\mathbf{S}^2 \otimes \Lambda^3\mathbf{S}^4 \subset \Lambda^5$ . Finally,  $\gamma$  sits in  $\Lambda^2\mathbf{S}^2 \otimes \Lambda^2\mathbf{S}^4 \subset \Lambda^4$ , whereas  $*\gamma \in \mathbf{S}^2 \otimes \Lambda^3\mathbf{S}^4 \subset \Lambda^4$ .

We restrict the study to  $\mathcal{G}$ -structures with invariant intrinsic torsion, so the exterior differential  $d : \Lambda^k \rightarrow \Lambda^{k+1}$  is a  $\mathcal{G}$ -invariant map. The six invariant forms must then define a subcomplex of deRham's complex,

$$\begin{array}{ccccc} \Lambda^3 T^*M & \xrightarrow{d} & \Lambda^4 T^*M & \xrightarrow{d} & \Lambda^5 T^*M \\ \uparrow & & \uparrow & & \uparrow \\ \text{Span}_{\mathbb{R}}\{\alpha, \beta\} & \xrightarrow{d} & \text{Span}_{\mathbb{R}}\{\gamma, *\gamma\} & \xrightarrow{d} & \text{Span}_{\mathbb{R}}\{*\alpha, *\beta\} \end{array}$$

and the restricted  $d$  on the second line is determined by linear maps between the spaces spanned by the invariant couples. Thus the pair  $(d\alpha, d\beta) \in 2\mathbb{R} \subset \Lambda^4$  must be such that

$$(d\alpha, d\beta) = (\gamma, *\gamma)A,$$

where  $A$  is a linear transformation acting on the right on the frame  $(\gamma, *\gamma)$  of  $\mathbb{R} \oplus \mathbb{R} \subset \Lambda^4$ . In a completely similar manner

$$(d\gamma, d*\gamma) = (*\alpha, *\beta)B,$$

$B$  being a linear map acting on  $(*\alpha, *\beta) \in 2\mathbb{R} \subset \Lambda^5$ .

As our invariant-torsion scenario does not allow for invariant forms of degree higher than five,  $*\alpha$  and  $*\beta$  are forced to be closed. The condition  $d^2 = 0$  becomes a non-linear constraint on the coefficients of  $A, B$

$$BA = \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the sequel we shall extract information on the torsion out of the defining matrices. To set the pace we provide the full proof of one intermediate result only, in preparation for Theorem 6.2, in order to show the type of computations and arguments involved. The key point to bear in mind is that, eventually, the invariant intrinsic torsion will depend linearly on the three terms

$$(6.1) \quad \begin{array}{ll} t_0 & \text{from } S^4 \otimes S^4 \subset T^* \otimes (\mathfrak{su}(3)/\mathfrak{g}), \\ t_+ & \text{from } \tau_{\mathcal{G}}^P(R) \in T^* \otimes (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))/\mathfrak{g}, \\ t_- & \text{from } \tau_{\mathcal{G}}^Q(R) \in T^* \otimes (\mathfrak{so}(3) \oplus \mathfrak{so}(5))/\mathfrak{g}. \end{array}$$

**Lemma 6.1.** *The coefficients  $b_2^1$  and  $a_1^1$  are proportional, and contain all the information provided by the invariant torsion in  $\tau_{\mathcal{G}}^P(Q) \in T^* \otimes (\mathfrak{su}(3)/\mathfrak{g})$ .*

*Proof.* The differential of an arbitrary  $\mathcal{G}$ -invariant form  $\Phi$  can be expressed by the action of the intrinsic torsion  $\tau$ , hence written as  $d\Phi = \tau \circ \Phi$ . Let  $\Pi_k(S^i \otimes S^j)$  indicate the projection to the submodule  $S^k$  in the Clebsch–Gordan expansion of  $S^i \otimes S^j$ .

We define a  $\mathcal{G}$ -equivariant mapping  $F_1 : \Lambda^3 S^2 \rightarrow \Lambda^2 S^2 \cdot \Lambda^2 S^4$  by skew-symmetrising the interior product of  $\Phi \in \Lambda^3 S^2$  with an element  $X \in \tau_{\mathcal{G}}^P(Q) = T^* \otimes (\mathfrak{su}(3)/\mathfrak{g})$

$$F_1(\Phi) := \mathcal{A}(X \lrcorner \Phi) \in \mathcal{A}(S^4 \otimes \Pi_4(S^4 \otimes S^2) \otimes S^2 \otimes S^2)$$

It is easy to see this is the unique map sending  $\Lambda^3 S^2$  to  $\Lambda^2 S^2 \cdot \Lambda^2 S^4$ . Moreover, Schur's Lemma guarantees it is a multiple of the identity map, that sends  $\alpha$  to  $\gamma$ . The proportionality factor is necessarily of the form  $f^\alpha t_0$ , where  $f^\alpha \in \mathbb{R} \setminus \{0\}$  and  $t_0$  is the  $\mathcal{G}$ -invariant

torsion component in  $S^4 \otimes S^4 \subset T^* \otimes (\mathfrak{su}(3)/\mathfrak{g})$

$$d\alpha = F_1(\alpha) = f^\alpha t_0 \gamma = a_1^1 \gamma.$$

On the same grounds there is a unique  $\mathcal{G}$ -equivariant function defining  $d : S^2 \Lambda^3 S^4 \rightarrow \Lambda^5 S^4$ , namely  $F_2 : S^2 \otimes S^4 \otimes S^4 \otimes S^4 \rightarrow \mathcal{A}(S^4 \otimes \Pi_4(S^4 \otimes S^2) \otimes S^4 \otimes S^4 \otimes S^4)$ . It involves contracting with the only component of the  $\mathcal{G}$ -invariant intrinsic torsion in  $\tau_{\mathcal{G}}^P(Q)$ , so again  $F_2$  maps  $*\gamma$  to  $*\alpha$  isomorphically

$$d(*\gamma) = f^\gamma t_0(*\alpha),$$

with  $f^\gamma \neq 0$ . Eventually,  $b_2^1 = f^\gamma t_0$  and the proof is complete. qed

We are in the position of generalising the previous discussion to all six  $\mathcal{G}$ -invariant forms, and define the following maps in analogy to  $F_1, F_2$

$$\begin{aligned} G_1(S^2 \otimes \dots) &= \mathcal{A}(S^2 \otimes \Pi_2(S^2 \otimes S^2) \otimes \dots), & H_1(S^2 \otimes \dots) &= \mathcal{A}(S^2 \otimes \Pi_4(S^2 \otimes S^2) \otimes \dots), \\ G_2(S^4 \otimes \dots) &= \mathcal{A}(S^2 \otimes \Pi_4(S^2 \otimes S^4) \otimes \dots), & H_2(S^2 \otimes \dots) &= \mathcal{A}(S^4 \otimes \Pi_2(S^4 \otimes S^2) \otimes \dots), \\ G_3(S^4 \otimes \dots) &= \mathcal{A}(S^4 \otimes \Pi_2(S^4 \otimes S^4) \otimes \dots), & H_3(S^4 \otimes \dots) &= \mathcal{A}(S^4 \otimes \Pi_4(S^4 \otimes S^4) \otimes \dots). \end{aligned}$$

The usual representation-theoretical argument gives

$$\begin{aligned} d\alpha &= F_1(\alpha) = f^\alpha t_0 \gamma; \\ d\beta &= (G_1 + G_2 + G_3 + H_1 + H_2 + H_3) \beta \\ &= \left( g_1^\beta(t_\pm) + g_2^\beta(t_\pm) + g_3^\beta t_0 \right) \gamma + \left( h_1^\beta(t_\pm) + (h_2^\beta + h_3^\beta) t_0 \right) *\gamma; \\ d\gamma &= (H_1 + H_2 + H_3) \gamma = (h_1^\gamma(t_\pm) + (h_2^\gamma + h_3^\gamma) t_0) *\beta; \\ d(*\gamma) &= (F_2 + G_1 + G_2 + G_3) *\gamma \\ &= (f^\gamma t_0) *\alpha + (g_1^\gamma(t_\pm) + g_2^\gamma(t_\pm) + g_3^\gamma t_0) *\beta, \end{aligned}$$

where  $t_0, t_+, t_-$  are as in (6.1). As for the rest,  $f^\alpha, g_3^\beta, h_2^\beta, h_3^\beta, h_2^\gamma, h_3^\gamma, g_3^\gamma, f^\gamma$  are constants, while the remaining  $g_1^\beta, g_2^\beta, h_1^\beta, h_1^\gamma, g_1^\gamma, g_2^\gamma$  are linear functions of  $t_\pm$ ; for example  $g_1^\beta(t_\pm) = g_1^{\beta-} t_- + g_1^{\beta+} t_+$ .

The entries of matrices  $A, B$  depend linearly on  $t_\pm, t_0$ ; we omit to write the explicit expressions merely for the sake of brevity. Simple computations produce other constraints, which will not be stated formally for the same reasons. Overall, though, the picture is that the  $\mathcal{G}$ -invariant intrinsic torsion is housed in

$$\begin{aligned} A &= \left[ \begin{array}{c|c} f^\alpha t_0 & (g_1^{\beta-} + g_2^{\beta-}) t_- - f^\alpha t_0 + (g_1^{\beta+} + g_2^{\beta+}) t_+ \\ \hline 0 & h_1^{\beta-} t_- + h_1^{\beta+} t_+ \end{array} \right] \\ B &= \left[ \begin{array}{c|c} 0 & m f^\alpha t_0 \\ \hline h_1^{\gamma-} t_- + (h_2^{\gamma-} + h_3^{\gamma-}) t_0 + h_1^{\gamma+} t_+ & (g_1^{\gamma-} + g_2^{\gamma-}) t_- + g_3^\gamma t_0 - h_1^{\gamma+} t_+ \end{array} \right] \end{aligned}$$

subject to linear constraints

$$\begin{aligned} f^\alpha h_1^{\beta\pm} &= f^\alpha h_1^{\gamma\pm} = f^\alpha (h_2^\gamma + h_3^\gamma) = 0, & (g_1^{\beta\pm} + g_2^{\beta\pm})(h_2^\gamma + h_3^\gamma) + h_1^{\beta\pm} g_3^\gamma &= 0, \\ (g_1^{\beta\pm} + g_2^{\beta\pm})h_1^{\gamma\pm} + h_1^{\beta\pm}(g_1^{\gamma\pm} + g_2^{\gamma\pm}) &= 0, & (g_1^{\beta\pm} + g_2^{\beta\pm})h_1^{\gamma\mp} + h_1^{\beta\pm}(g_1^{\gamma\mp} + g_2^{\gamma\mp}) &= 0, \end{aligned}$$

where  $h_1^{\gamma+}$  is proportional to  $(g_1^{\gamma+} + g_2^{\gamma+})$ .

To sum-up,

**Theorem 6.2.** *Let  $\{M^8, g\}$  be a  $\mathcal{G}$ -manifold equipped with the six  $\mathcal{G}$ -invariant forms  $\{\alpha, \beta, \gamma, *\gamma, *\alpha, *\beta\}$ . If the intrinsic torsion is  $\mathcal{G}$ -invariant, the differential forms satisfy one of the following four sets of differential equations*

	$d\alpha$	$d\beta$	$d\gamma$	$d(*\gamma)$	$d(*\alpha)$	$d(*\beta)$
I	$a_1^1 \gamma$	$a_2^1 \gamma$	0	$ma_1^1(*\alpha) + b_2^2(*\beta)$	0	0
II	0	$a_2^1 \gamma + a_2^2(*\gamma)$	$b_1^2(*\beta)$	$-((a_2^1 b_1^2)/a_2^2)(* \beta)$	0	0
III	0	$a_2^1 \gamma$	0	$b_2^2(*\beta)$	0	0
IV	0	0	$b_1^2(*\beta)$	$b_2^2(*\beta)$	0	0

*Proof.* The aforementioned constraints on  $A, B$  can be recast by the more elegant

$$(6.2) \quad b_2^1 a_2^2 = ma_1^1 a_2^2 = b_1^2 a_1^1 = 0$$

$$(6.3) \quad b_1^2 a_2^1 + b_2^2 a_2^2 = 0$$

for some real  $m$ , and four generic cases ought to be considered.

I. Suppose  $a_1^1 \neq 0$ , so  $b_2^1 = ka_1^1 \neq 0$ ; in order to satisfy equations (6.2) we have to impose  $a_2^2 = b_1^2 = 0$ . Then (6.3) holds necessarily, for any  $a_2^1$ ,  $b_2^2$ , and the ranks  $r_A, r_B$  are both equal to 1. At the level of forms,

$$d\alpha = a_1^1 \gamma, \quad d\beta = a_2^1 \gamma, \quad d\gamma = 0, \quad d(*\gamma) = ma_1^1(*\alpha) + b_2^2(*\beta).$$

II. Assume  $a_1^1 = 0$ , so that  $b_2^1 = 0$ , and we are left with (6.3) only. Supposing  $a_2^2 \neq 0$  we obtain an extra relation implying  $b_2^2 = -(a_2^1 b_1^2)/a_2^2$  for arbitrary  $a_2^1$ ,  $b_1^2$ . The rank of  $A$  is one, while  $r_B = 0$  or 1 depending on  $b_1^2$ . Therefore

$$d\alpha = 0, \quad d\beta = a_2^1 \gamma + a_2^2(*\gamma), \quad d\gamma = b_1^2(*\beta), \quad d(*\gamma) = -\frac{a_2^1 b_1^2}{a_2^2}(*\beta).$$

III. For  $a_1^1 = 0$  and  $a_2^2 = 0$ , we are left with  $b_1^2 a_2^1 = 0$ . Taking  $a_2^1 \neq 0$  forces  $b_1^2 = 0$ , and  $b_2^2$  is free. Again,  $r_A = 1$ , and  $b_2^2$  decides whether  $r_B = 0$  or 1. The forms satisfy

$$d\alpha = 0, \quad d\beta = a_2^1 \gamma, \quad d\gamma = 0, \quad d(*\gamma) = b_2^2(*\beta).$$

IV. If in the previous case we assume  $a_2^1 = 0$  then  $b_1^2$  becomes a free parameter, as does  $b_2^2$ . Now  $A$  is null and  $0 \leq r_B \leq 1$  depending on  $b_1^2, b_2^2$ . Hence

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = b_1^2(*\beta), \quad d(*\gamma) = b_2^2(*\beta).$$

qed



One fact deserves an explanation. Since the quaternionic 4-form  $\Omega$  of Kraines and Bonan [19, 5] can be decomposed as

$$\Omega = \gamma + *\gamma$$

using a suitable four-form  $\gamma$ , and its derivative  $d(\gamma + *\gamma) = b_2^1(*\alpha) + (b_1^2 + b_2^2)(* \beta)$  ‘is’ the quaternionic torsion  $\tau_{\mathrm{Sp}(2)\mathrm{Sp}(1)}$  in disguise, one can easily see that the coefficients are linear in  $\{t_-, t_0\}$  only, as  $t_+$  does not appear in the invariant relative  $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -torsion.

On the other hand there exists a second 4-form

$$(6.4) \quad \Omega' = \gamma - *\gamma$$

with stabiliser  $\mathrm{Sp}(2)\mathrm{Sp}(1) \subset \mathrm{GL}(2, \mathbb{H})\mathbb{H}^*$ , that arises by changing the orientation of the almost quaternion-Hermitian structure. Unlike  $\Omega$ , the anti-selfdual  $\Omega'$  is a function of  $t_0$  and of a linear combination of both  $t_+, t_-$ . The presence of two (non-conjugated) Lie groups isomorphic to  $\mathrm{Sp}(2)\mathrm{Sp}(1)$  is consistent with Remark 3.1.

**Examples 6.3.** *To finish, we indicate how to find examples befitting the Theorem’s cases.*

I: *Both  $A$  and  $B$  have rank 1, so we can take  $M = \mathrm{SU}(3)$  as in example 2.2.*

II: *While  $A$  has rank 1, for  $B$  it is either 0 or 1. Since the quaternionic form  $\Omega$  is closed iff  $b_1^2 + b_2^2 = 0$ , necessarily  $\Omega = d\beta$  is exact, and many constructions are known, see [31, 27].*

III: *Suppose we insist on wanting  $\Omega = \gamma + *\gamma$  closed. As  $\gamma$  is always exact, we must force  $B$  to be null and therefore there are no  $\mathcal{G}$ -invariant four-forms.*

IV: *Invariant three-forms are closed ( $A = 0$ ), and [13, 24] provide us with constructed through special foliations.*

*Further examples are easy to build, and the previous computations indicate that a classification of sorts is within sight. The 5-dimensional theory has the advantage of highlighting the prominent geometrical aspects of dimension 8, and one can take, for instance, a rank-three bundle over one explicit 5-dimensional Lie group of [10], or use Cartan-Kähler techniques as in the last section of [6]. The instances of [2] suggest nice twistor-flavoured constructions of similar type.*

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1) DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY

E-mail address: oscarmacia@calvino.polito.it

E-mail address: simon.chiossi@polito.it

2) DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE VALENCIA, C. DR. MOLINER,  
S/N, 96100 BURJASSOT, VALENCIA, SPAIN

*E-mail address:* `oscar.macia@uv.es`